Translating observed (Q,U)-maps to B-mode power spectrum and scientific results

------ Constructing the optimal estimators

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B-modes are not observables

- **For the linear polarized photon fields, the observables are Q-map and U-map in 2-d sphere.**

- **Unfortunately, they are NOT scalar fields.**

- **In general, we deal with them as follows:** $P_{\pm}(\hat{\gamma}) \equiv Q(\hat{\gamma}) \pm iU(\hat{\gamma}) = \sum_{\ell m} a_{\pm 2,\ell m} \; {}_{\pm 2}Y_{\ell m}(\hat{\gamma}),$

- **The scalar B-mode are defined as (similar for E-mode):**

$$B_{\ell m} \equiv -\frac{1}{2i}\left[a_{2,\ell m} - a_{-2,\ell m}\right] \quad B(\hat{\gamma}) \equiv \sum_{\ell m} B_{\ell m} Y_{\ell m}(\hat{\gamma}). \quad C_\ell^{BB} \equiv \frac{1}{2\ell+1}\sum_m \langle B_{\ell m} B^{\star}_{\ell m}\rangle,$$

- **In full-sky case, the best-unbiased estimators are given by**

$$\hat{C}_\ell^{BB} = \frac{1}{2\ell+1}\sum_{m=-\ell}^{\ell} B_{\ell m} B^{\star}_{\ell m}, \quad \langle \Delta\hat{C}_\ell^{BB} \Delta\hat{C}_{\ell'}^{BB}\rangle = \frac{2\delta_{\ell\ell'}}{2\ell+1}(C_\ell^{BB})^2,$$

QML estimator based on (Q,U)-maps

- **In (Tegmark & Oliveira-Costa, 2001), Tegmark extended the Quadratic Maximal Likelihood (QML) method in T-map to the polarization case, which are defined as follows:**

$\mathbf{x} \equiv (\mathbf{\Delta T}, \mathbf{Q}, \mathbf{U})$.

The optimal QML power spectrum estimate is (TdO01)

$$y_\ell^r = x_i x_j E_{ij}^{r\ell}, \qquad r \equiv (T, X, E, B),$$

where the matrices $E^{r\ell}$ are given by

$$E^{r\ell} = \frac{1}{2} C^{-1} \frac{\partial C}{\partial C_\ell^r} C^{-1},$$

and C is the covariance matrix of the data vector $\mathbf{x}$,

$$C_{ij} = \langle x_i x_j \rangle = \begin{pmatrix} C^{TT} & C^{TQ} & C^{TU} \\ C^{QT} & C^{QQ} & C^{QU} \\ C^{UT} & C^{UQ} & C^{UU} \end{pmatrix}.$$

$$\langle y_\ell^r \rangle = F_{\ell\ell'}^{sr} C_{\ell'}^s,$$

where

$$F_{\ell\ell'}^{sr} = \frac{1}{2} \mathrm{Tr} \left[\frac{\partial C}{\partial C_{\ell'}^s} C^{-1} \frac{\partial C}{\partial C_\ell^r} C^{-1} \right].$$

$$\langle y_\ell^r y_{\ell'}^s \rangle - \langle y_\ell^r \rangle \langle y_{\ell'}^s \rangle \equiv F_{\ell\ell'}^{rs} = 2\mathrm{Tr} \left[C E^{r\ell} C E^{s\ell'} \right],$$

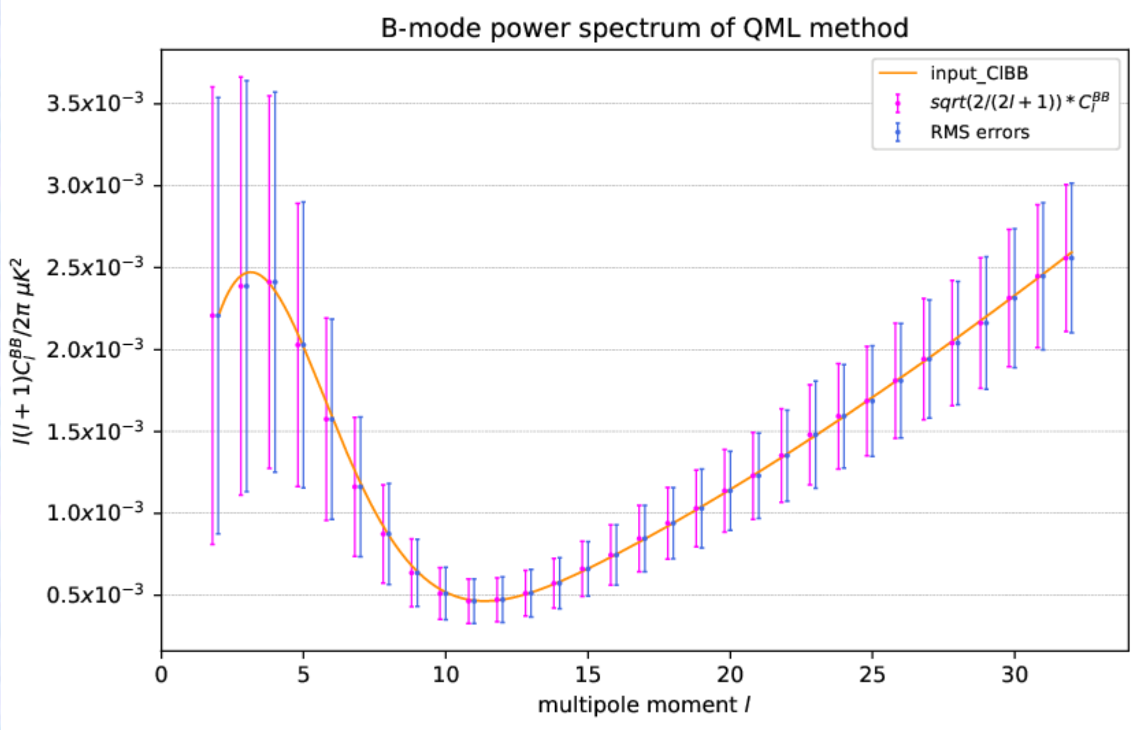
- **The authors proved that they are the best(minimal error bars) one.**

$$\hat{C}_\ell^r = F^{-1} y. \qquad \langle \Delta\hat{C}_\ell \Delta\hat{C}_{\ell'} \rangle = F^{-1}$$

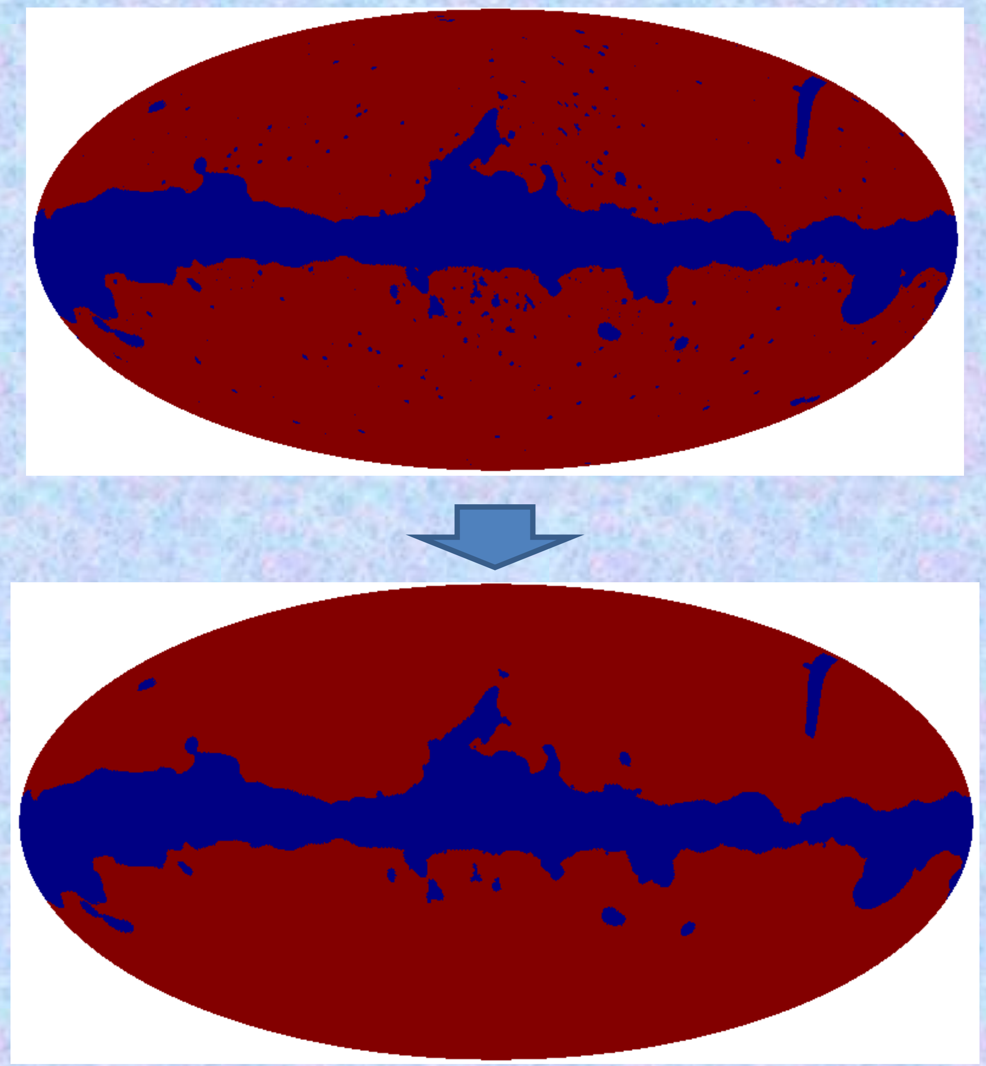
QML estimator based on (Q,U)-maps

- **For polarization, it includes the inversion and multiplication of $(2N_d)X(2N_d)$ matrices, so it require an operations with order of $(2N_d)^3$.**

- **In the ideal case, we consider the full-sky case, noise-free, l_{max}=32, N_{side}=16 in simulations. We recover the results of cosmic variance limit.**



QML estimator in Planck mask



QML estimator in Planck mask

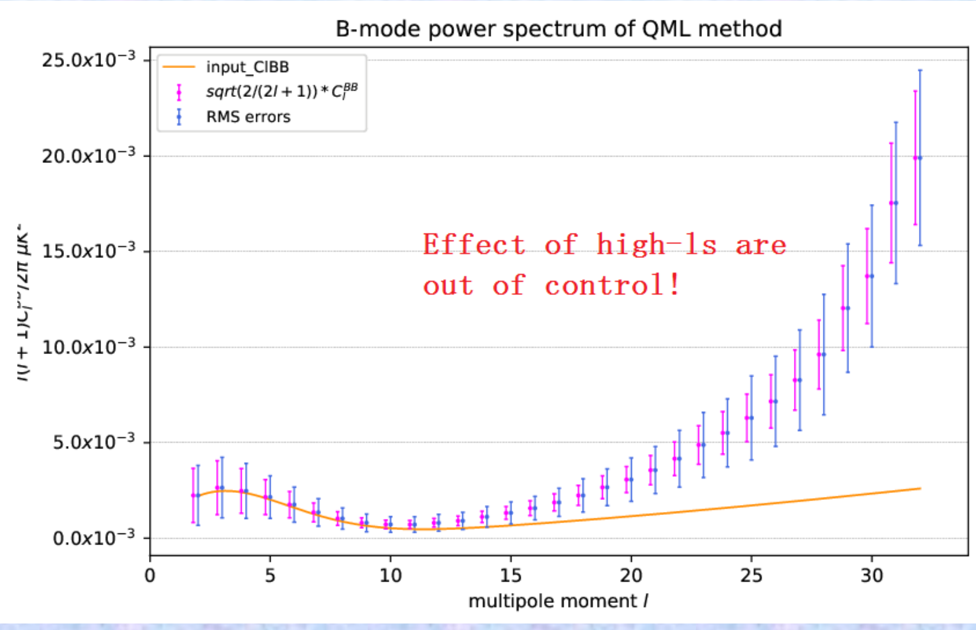
- **First: Comparing with ideal case, we should degrade the high-resolution map to the low-resolution map.**

- **In principle, which may cause E-B mixture. But, by simulation, we find this mixture is negligible.**

- **Second: In practice, in masked sky, the multiple mixture is quite very important, since in this method, the mixture occurs between:**

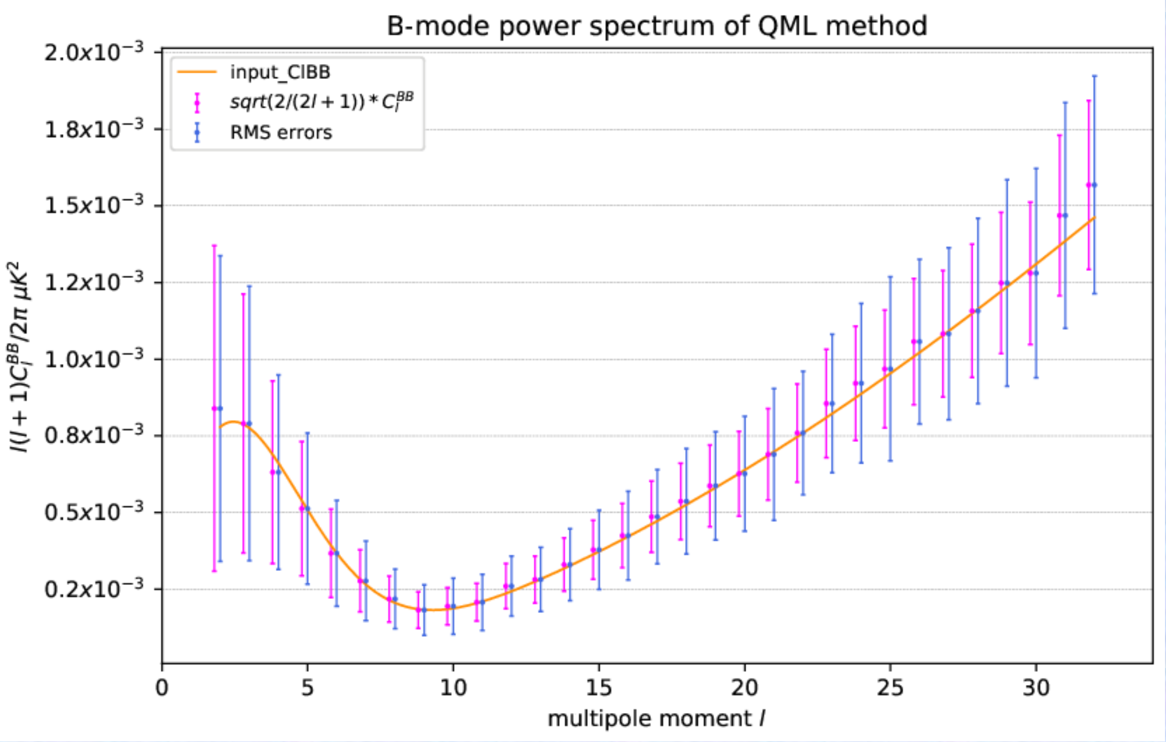
$$l^4 C_l^{EE\,(BB)} \Leftrightarrow l'^4 C_{l'}^{EE\,(BB)}$$

Very Blue Spectrum !!!



QML estimator in Planck mask

- **To reduce it, we should smooth the map in advance, and then apply to QML method.**
- **The smoothing scale should be chosen carefully.**



E-B separation and Pure B-mode

- **B-mode is not the local quantity, so E-B mixture exists in general.**
- **To remove E-B mixture, we can consider another definition of E-, B-mode polarization as follows [Zhao & Baskaran, 2010]:**

$$\mathcal{E}(\hat{\gamma}) \equiv -\frac{1}{2}\left[\bar{\eth}_1\bar{\eth}_2 P_+(\hat{\gamma}) + \eth_1\eth_2 P_-(\hat{\gamma})\right],$$
$$\mathcal{B}(\hat{\gamma}) \equiv -\frac{1}{2i}\left[\bar{\eth}_1\bar{\eth}_2 P_+(\hat{\gamma}) - \eth_1\eth_2 P_-(\hat{\gamma})\right],$$

which relates to the general B-mode by $\mathcal{B}_{\ell m} = N_\ell B_{\ell m}$. $N_\ell \equiv \sqrt{(\ell+2)!/(\ell-2)!}$

- **In practice, the masked map can be constructed as follows, which is a scalar field, and free from E-B leakage,**

$$\mathcal{B}(\hat{\gamma})W(\hat{\gamma}) = \sum_{\ell m} \tilde{\mathcal{B}}_{\ell m} Y_{\ell m}(\hat{\gamma}),$$

where [Smith & Zaldarriaga (SZ), 2007]:

$$\tilde{B}_{\ell m} = \frac{i}{2}\int d\hat{\gamma} P(\hat{\gamma})\left[W(\hat{\gamma})({}_2Y^*_{\ell m}(\hat{\gamma})) + \frac{2W_1^*(\hat{\gamma})({}_1Y^*_{\ell m}(\hat{\gamma}(\hat{\gamma})))}{\sqrt{(\ell-1)(\ell+2)}} + \frac{W_2^*(\hat{\gamma})Y^*_{\ell m}}{\sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}}\right] + \text{c.c.}$$

“QML+SZ” estimator

- **The constructed B-mode map is a scalar field. Thus, the QML estimator in the Temp-map can be directly applied here, which is given by:**

$$\hat{C}_\ell^{BB} = N_\ell^{-2}\hat{C}_\ell^{\mathcal{B}\mathcal{B}} = N_\ell^{-2}\sum_{\ell'}(F^{-1})_{\ell\ell'}y_{\ell'}$$

$$F_{\ell\ell'} = \frac{1}{2}\mathrm{Tr}\left[C^{-1}\frac{\partial C}{\partial C_\ell^{\mathcal{B}\mathcal{B}}}C^{-1}\frac{\partial C}{\partial C_{\ell'}^{\mathcal{B}\mathcal{B}}}\right].$$

$$y_\ell = \sum_{i,j=1}^{N_{\rm pix}}\mathcal{B}_i\mathcal{B}_j\Theta_{ij}^\ell,\ \ \Theta_{ij}^\ell = \frac{1}{2}\sum_{k,k'}(C^{-1})_{ij}\frac{\partial C_{kk'}}{\partial C_\ell^{\mathcal{B}\mathcal{B}}}(C^{-1})_{k'j}$$

- **The covariance matrix is given by:**

$$\langle\Delta\hat{C}_\ell^{BB}\Delta\hat{C}_{\ell'}^{BB}\rangle = N_\ell^{-4}\langle\Delta\hat{C}_\ell^{\mathcal{B}\mathcal{B}}\Delta\hat{C}_{\ell'}^{\mathcal{B}\mathcal{B}}\rangle = N_\ell^{-4}(F^{-1})_{\ell\ell'}$$

which is the Cramer-Rao bound.

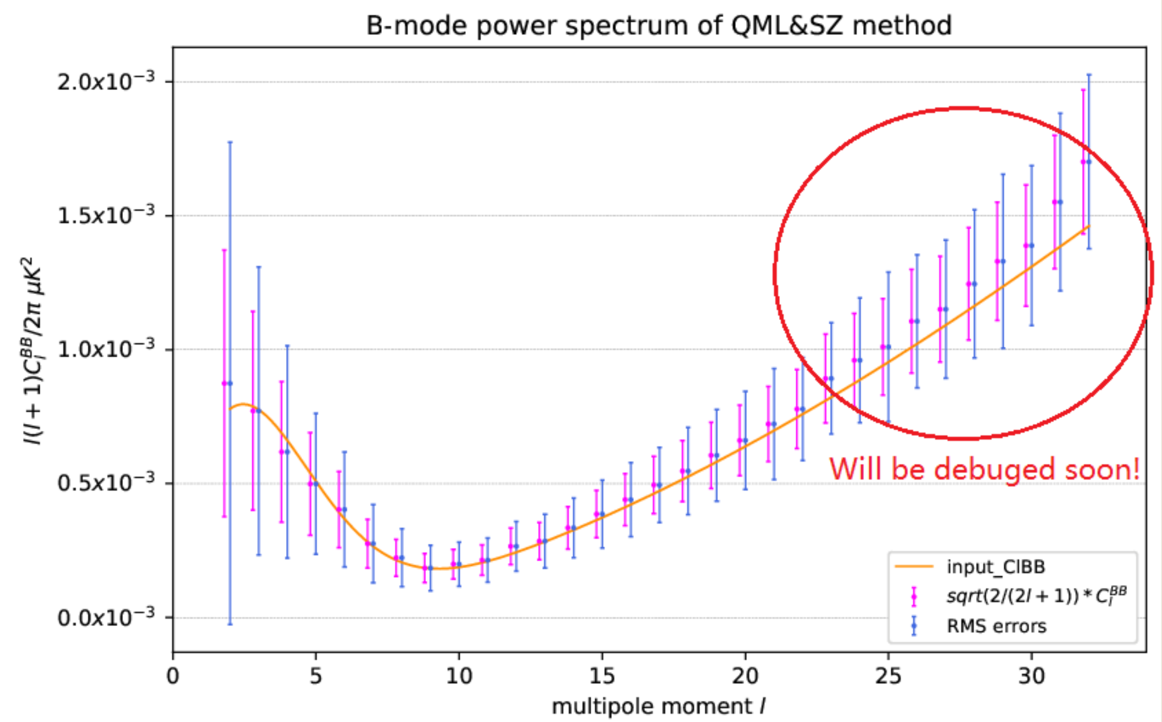
- **It includes the inversion and multiplication of $N_d X N_d$ matrices, so it require an operations with order of N_d^3, which is “one order” smaller than the traditional QML.**

“QML+SZ” estimator in Planck mask

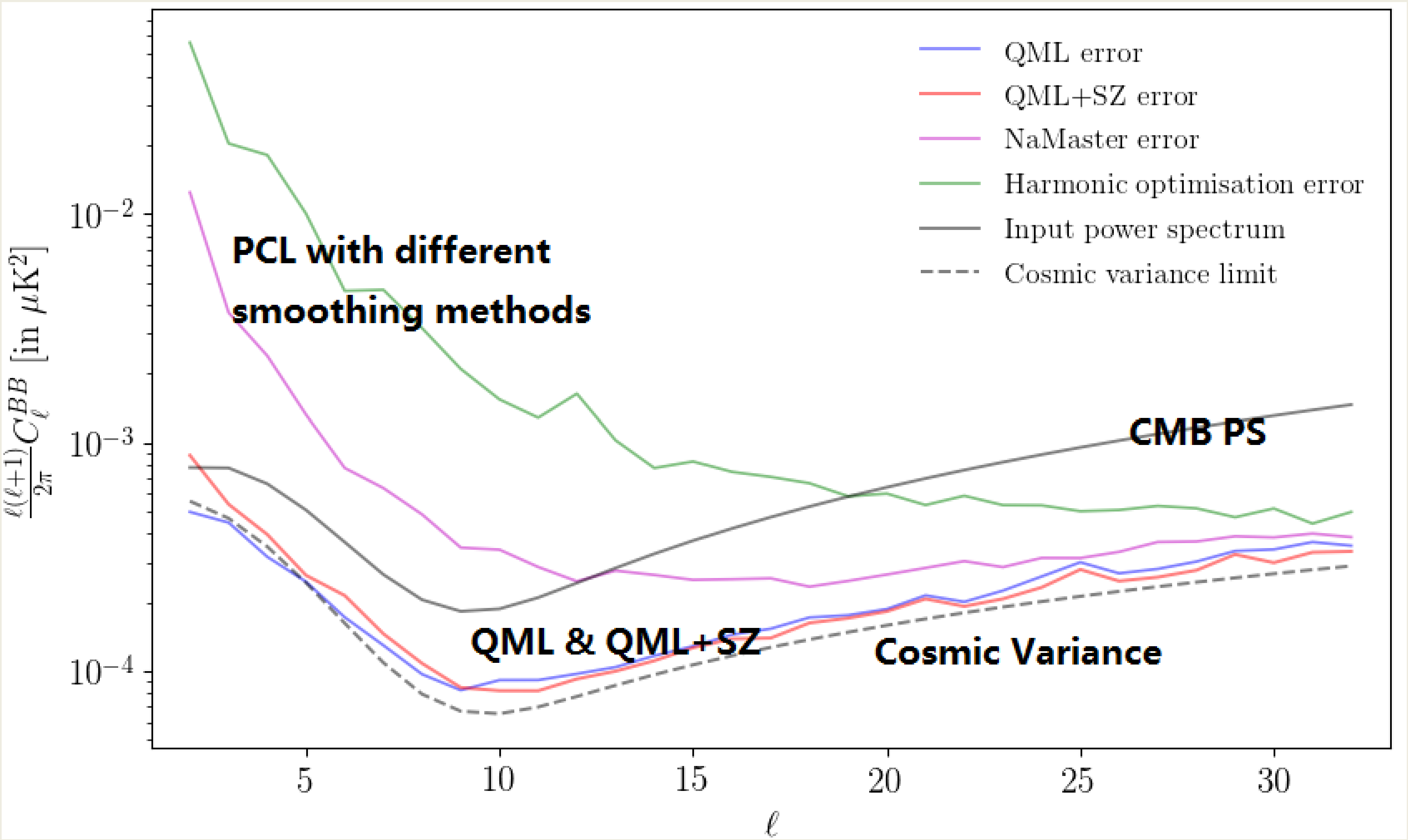
- **Similarly, we should smooth the constructed B-map in advance to reduce the effect of high-ls. The mixture occurs between:**

$$l^4 C_l^{BB} \Leftrightarrow l'^4 C_{l'}^{BB}$$ **Very Blue Spectrum !!!**

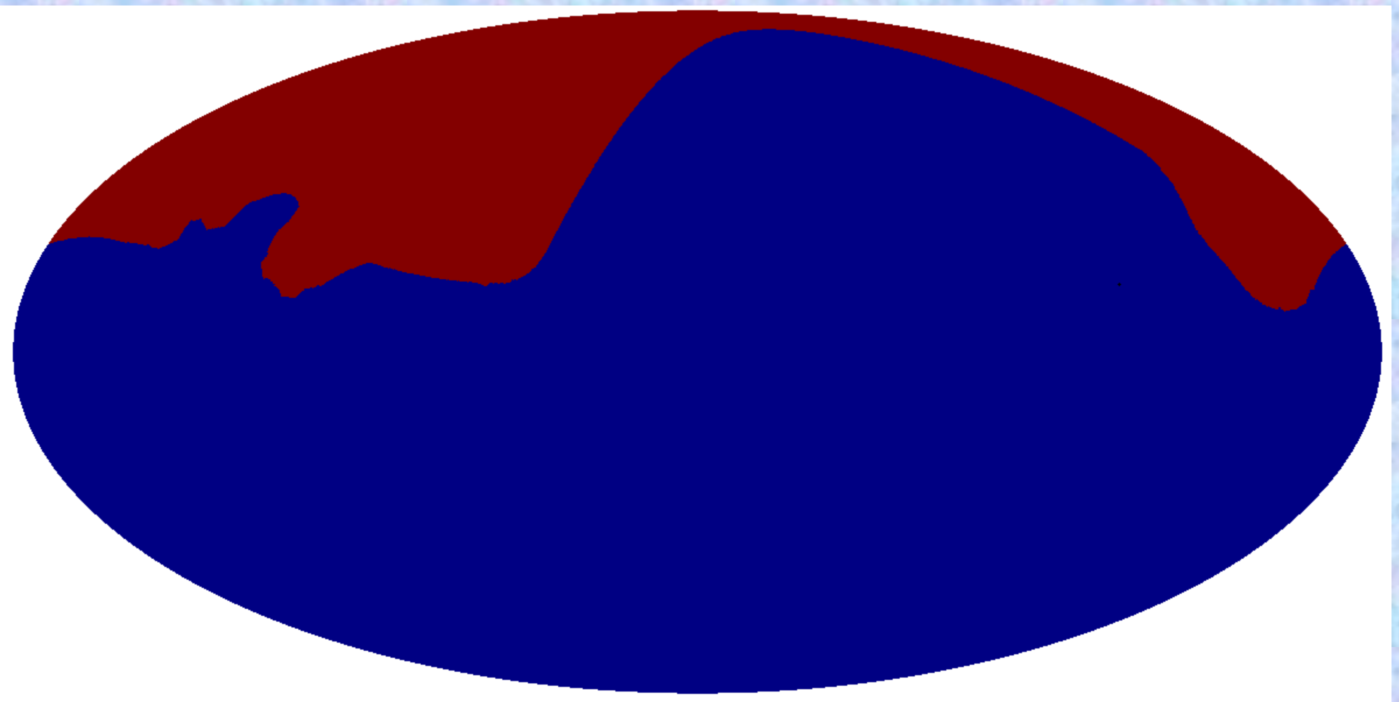
- **By simulation, we properly choose the smoothing method and obtain the results:**



Comparison of different estimators

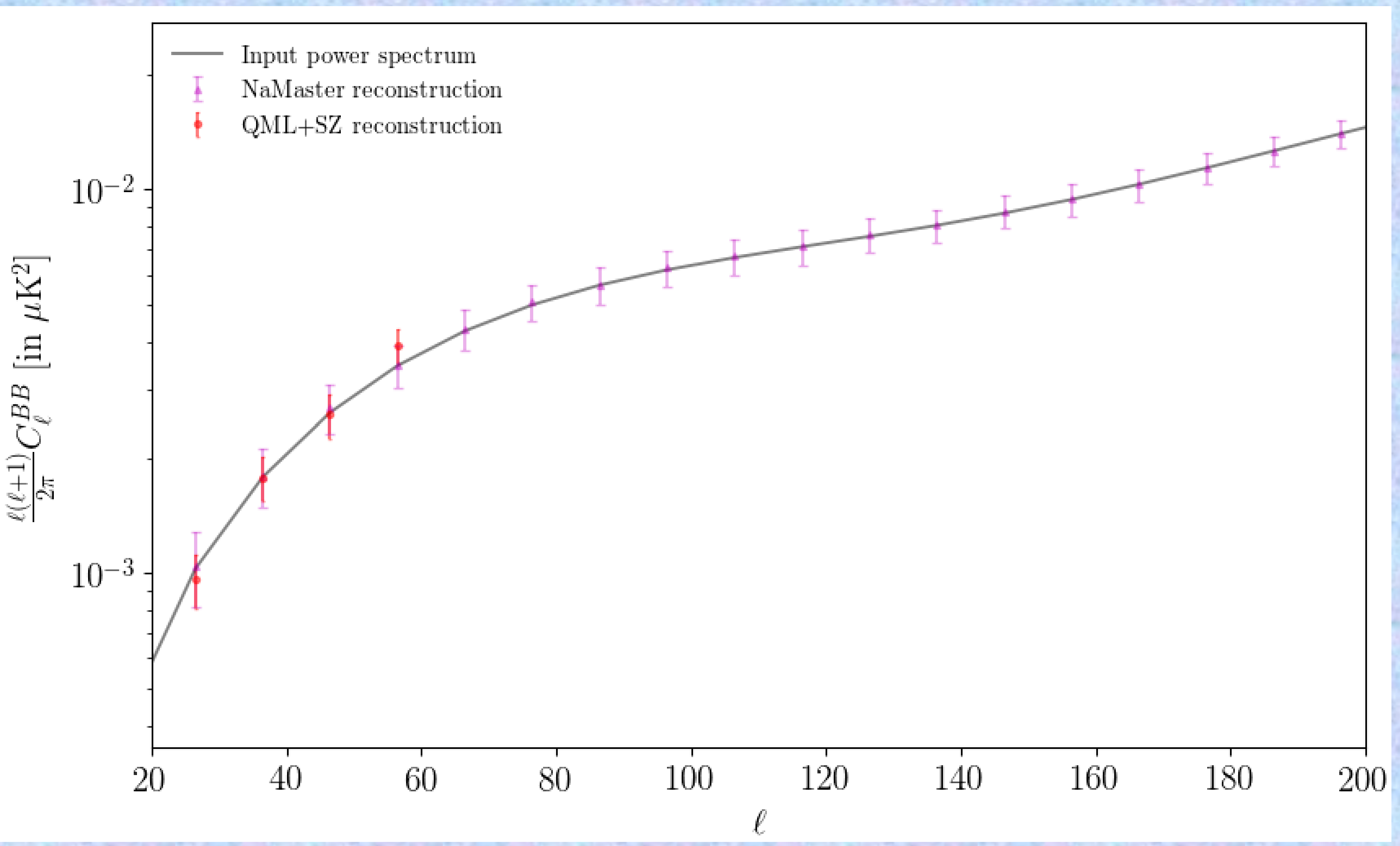


Applying to AliCPT mask



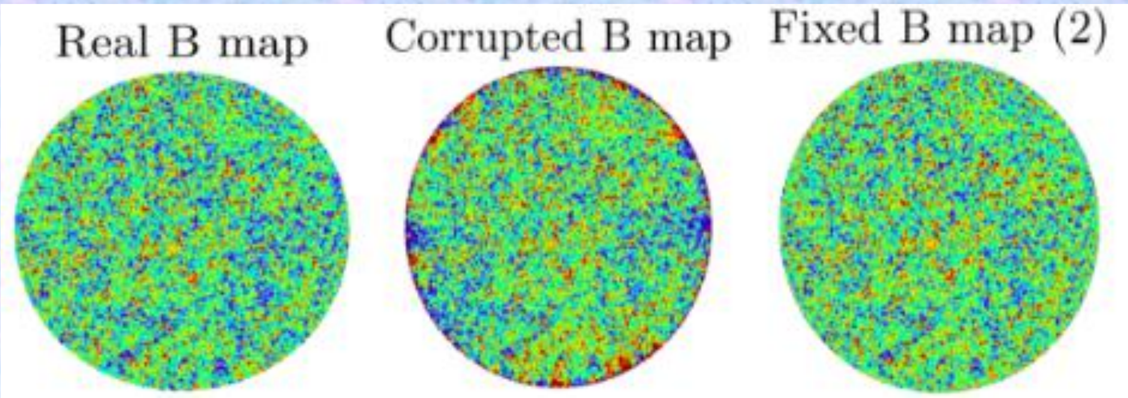
“f_{sky}=0.1”, QML or QML+SZ can be applied for “l<128”

Applying to AliCPT mask



Work Plans

- **Extend QML methods to the cases with inhomogeneous noise.**
- **QML (low-l) + PCL (high-l) → Hybrid estimators for BB, EB, TB** [Efstathiou, 2003, 2006]**, and apply to AliCPT.**
- **Develop the hybrid estimator based on the optimal B-map** [Hao Liu, 2018, 2019]**, which is suspected to be free from the heavy leakage from high-ls to low-ls.**



- Thanks!